CASTIGLIANO THEOREM

ENERGY METHOD

Overview of Loads ON and IN Structures / Machines

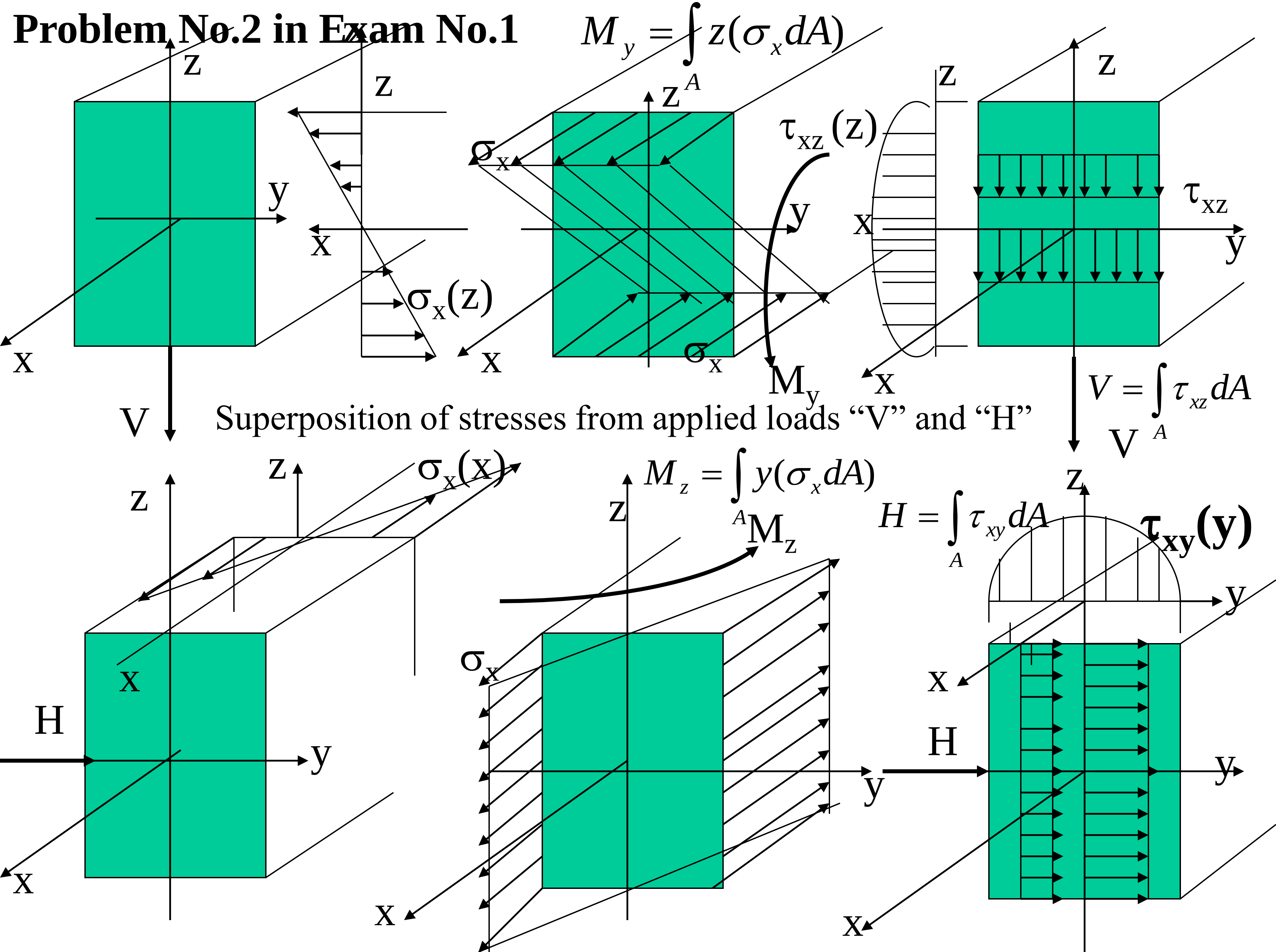
- **Structural Analysis**
  - **Applied Loads- Forces &Moments**
    - Body forces &couples
    - Surface forces&couples
      - Concentrated
      - Distributed / Pressures
  - **Reaction Forces & Moments (at supports)**
    - From FBD of entire structure
      - Flow Lines from Applied to Reaction Forces
      - FBD of Each Component from Forces through Contact Surfaces
    - Equilibrium Eqs. (3 in 2D, 6 in 3D)
  - **Internal Forces & Moments**
    - Resultants on Cutting Surface
    - From FBD of Part of Structure
    - Equilibrium Eqs. (3 in 2D, 6 in 3D)
    - Depend on Location& Orientation of Cutting Plane
      - Shear Force & Bending Moment Diagrams

Overview of Various Stress Patterns

Stresses- from Distribution of Internal Forces over Given Cross-Section

- **Uniaxial Tension or Compression of Straight Bars or Lap Joints**
  - Uniform Distributions of Stresses
  - Normal Stress-Straight Bars Sigma=F/A
  - Direct Shear in Lap Joints Tau =P/A
- **Pure Bending of Long Beams**
  - Neutral Axis at Centroid of Cross-section
  - Linear Distribution over from Neutral Axis Sigma=(My)/I (Normal Axial Stresses)
    - The Resultant of Bending Stresses is the Bending Moment in the Cross-section
    - Maximum at top or bottom, and zero at neutral axis
- **Beams Subjected to Transverse Forces**
  - Transverse Shear Stresses Result (Add-up) in Shear Force in the Cross-Section
  - Distribution from Neutral Axis Depends on Shape of Cross-section Tau(y)=(V*MomentArea)/(I*Z(y))
  - Maximum may not be at neutral axis Tau_max=(coef.)Tau_average
    - Zero at top and bottom edges
  - Bending of Asymmetric Beams: IF Plane of Transverse Forces passes through the SHEAR CENTER, NO Torsion occurs
- **Torsion of Round or Prismatic Shafts**
  - Moment of differential torsional stress about centroid results in the internal torque in cross-section
  - Circular Cross-section: Linear distribution, with Max. at the outer edge: Tau=(Tr)/J
  - Prismatic Shafts -Torsion Leads to Warping Membrane Analogy: Tau_max=T/Q, Theta=(TL)/KG
  - Power-Torque-RPM Relations: hp=(Tn)/63,025 kw=(Tn)/9549

Problem No.2 in Exam No.1



Superposition of stresses from applied loads “V” and “H”

Elastic Deformation for Different Types of Loadings (Stress Pattern

- **Straight uniform elastic bar loaded by centered axial force (Figs. 2.1, 2.2, 2.3)**
  - Similar to linear spring in elastic range

$$F = ky, \quad y = \delta_f = l_f - l_0$$

  - "Force-induced elastic deformation", δ_f must not exceed "design allowable": failure is predicted to occur if (FIPTOI): $\delta_{f\text{-max}} > \delta_{f\text{-allow}}$
  - "Spring Constant (Rate)" for elastic bar

$$k(lb/in) = F/y = F/\delta_f$$

  - Normalize "Force-deflection" curve to obtai "Engineering Stress-Strain Diagram"

$$\sigma = \frac{F}{A_0}, \quad \varepsilon_f = \frac{\delta_f}{l_0}, \quad so\,that:$$

$$k_{ax} = \frac{F}{\delta_f} = \frac{A_0\sigma}{l_0\varepsilon_f} = \frac{A_0 E}{l_0}, \quad where: \quad \sigma = E\varepsilon_f \ (Hooke's\ Law)$$

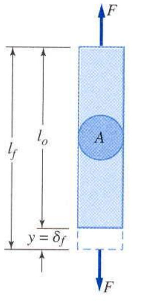


Figure 2.1

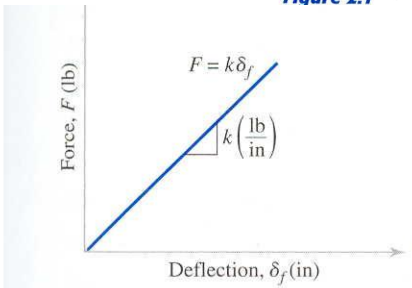


Figure 2.2
Force-deflection curve for linear elastic behavior

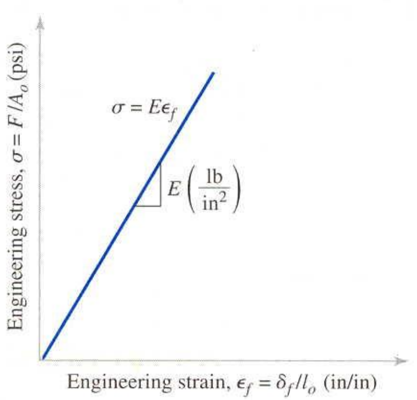


Figure 2.3
Engineering stress-strain diagram for linear elastic behavior.

Elastic Deformation for Different Types of Loadings (Stress Patterns)

- **Torsional moment produces torsional shearing strain, according to Hooke's Law: $\tau = G\,\gamma$**
  - Shear strain, γ, is change in initially right angle (radians)
  - Angular deflection (twist angle) for elastic members:

$$\theta = \frac{TL}{KG}, \quad where: \quad K = J \;\; for\ cylindrical\ members$$

- **Beam bending loads cause transverse deflections:**
  - Deflection (elastic) curve obtained by integrating twice the governing differential equation and using the boundary conditions:

$$\frac{d^2 y(x)}{dx^2} = \frac{M_z(x)}{EI_{zz}}, \; where\ "M"\ from\ bending\ moment\ diagram$$

  - See Table 4.1 for deflection curves of several common cases

Stored Strain Energy (Potential Energy of Strain)

- **From <u>Work</u> done by external forces or moments over corresponding displacements**
  - Recovered by gradual unloading if <u>elastic limit</u> of the material is not exceeded
  - Displacements (deformations) are LINEAR functions of external loads if <u>Hooke's Law</u> applies
  - <u>Generalized</u> forces include moments, and generalized displacements include rotations (angular displacements)
- **Strain energy <u>per unit volume</u> for differential cubic element (Fig. 4.11)**

$$u = \frac{U}{dxdydz} = \frac{\sigma_x \varepsilon_x}{2} + \frac{\sigma_y \varepsilon_y}{2} + \frac{\sigma_z \varepsilon_z}{2}$$

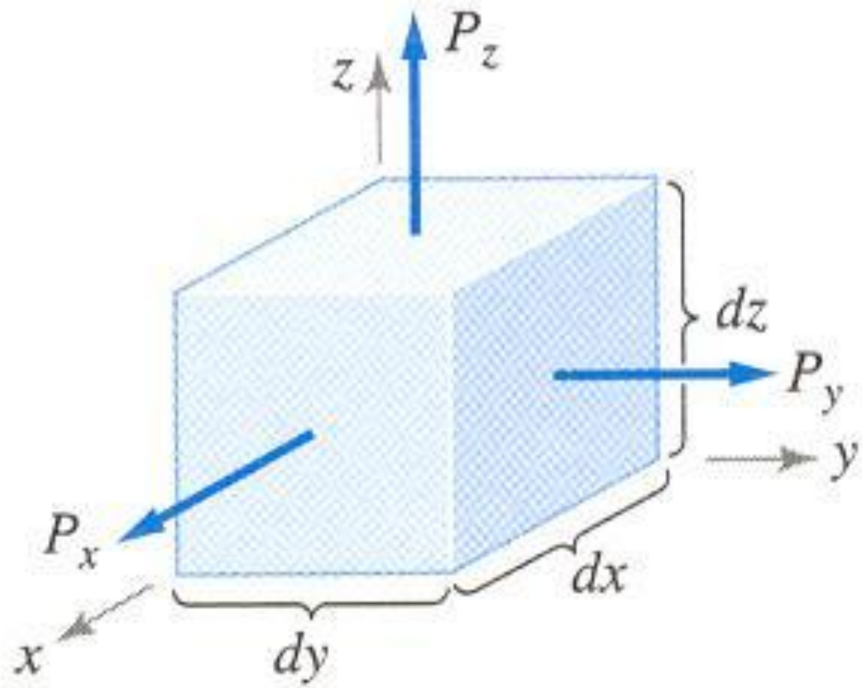


Figure 4.11

Unit cube of material subjected to mutually perpendicular forces P_x, P_y, and P_z.

Total Strain Energy Formulas for Common Stress Patterns

- **Members with <u>uniform (constant) geometry</u> material properties along the longitudinal axis**
  - Tension and Direct Shear

$$U_{tens} = F_{ave}\delta_f = \frac{F}{2}(\frac{F}{k_{ax}}) = \frac{F}{2}(\frac{FL}{AE}) = \frac{F^2L}{2AE}, \textit{ where A is cross-sectional area}$$

$$U_{dir-shear} = P_{ave}\delta_s = \frac{P}{2}(\frac{P}{k_{dir-shear}}) = \frac{P}{2}(\frac{PL}{AG}) = \frac{P^2L}{2AG}, \textit{ where A and L refer to the contact surface}$$

  - Torsion

$$U_{tor} = T_{ave}\theta_f = \frac{T}{2}(\frac{T}{k_{tor}}) = \frac{T}{2}(\frac{TL}{KG}) = \frac{T^2L}{2KG}$$

  - Pure bending

$$U_{bend} = M_{ave}\delta_{yf} = \frac{M}{2}(\frac{M}{k_{bend}}) = \frac{M}{2}(\frac{ML}{EI}) = \frac{M^2L}{2EI}$$

  - Strain energy associated with <u>transverse shearing stresses</u> is complex function of cross-section and negligible in comparison to bending strain energy (except for <u>short</u> beams)
- **Integrations are required if geometry or material properties <u>vary</u> along the member (Table 4.6, where "Q" denotes generalized displacement)**

TABLE 4.6 Summary of Strain Energy Equations and Deflection Equations for Use with Castigliano's Method Under Several Common Loading Conditions

Type of Load	Strain Energy Equation	Deflection Equation
Axial	$U = \int_0^L \frac{P^2}{2EA} dx$	$\delta = \int_0^L \frac{P(\partial P/\partial Q)}{EA} dx$
Bending	$U = \int_0^L \frac{M^2}{2EI} dx$	$\delta = \int_0^L \frac{M(\partial M/\partial Q)}{EI} dx$
Torsion	$U = \int_0^L \frac{T^2}{2KG} dx$	$\delta = \int_0^L \frac{T(\partial T/\partial Q)}{KG} dx$
Direct shear[1]	$U = \int_0^L \frac{P^2}{2AG} dx$	$\delta = \int_0^L \frac{P(\partial P/\partial Q)}{AG} dx$

[1]Transverse shear associated with bending gives a similar but more complicated function of the particular cross-sectional shape (see Figure 4.6 and associated discussion) and is usually negligible compared to the strain energy of bending.

Castigliano's Theorem

- **Energy method for calculating displacements in a deformed elastic body (Deflection equations - Table 4.6)**
  - At ANY point where an external force is applied, the displacement in the direction of that force is given by the partial derivative of the total strain energy with respect to that force.
  - Example of simple tension in uniform prismatic bar:

$$\frac{dU}{dF} = \frac{d}{dF}\left(\frac{F^2 L}{2EA}\right) = \frac{FL}{EA} = \delta_f$$

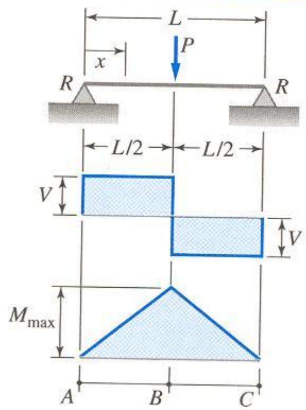
  - If no real force is applied at the point of interest, a "DUMMY" force is "applied" at that point, and then set equal to zero in the expression of the corresponding derivative of the total strain energy.
- **Applicable also to calculating reactions at the supports of statically redundant (undetermined) structures.**
  - Set partial derivative equal to zero since there is no displacement

Summary of Example Problems

- **Example 4.6 – Total strain energy in beam**
  - Simply supported beam loaded by concentrated load, “P” at mid-span and moment “M” at left support
  - Superposition of cases 1 and 4 in Table 4.1
- **Example 4.7 – Beam deflections and slopes**
  - Determine reactions and expressions of the bending moment on both sides of mid-span (from equilibrium)
  - Use deflection equations in Table 4.6 for mid-span deflection and angular displacement (slope of deflection curve) at left support
  - Apply dummy moment at mid-span to calculate angular deflection (slope of deflection curve) at mid-span

TABLE 4.1 Loading (P), Shear (V), and Moment (M) Diagrams for Selected Beam Configurations. Note t y is transverse deflection and θ is slope.

Case 1. Simple Beam; Concentrated Load P at Center



$$R = V = \frac{P}{2}$$

$$M_{max} \text{ (at point of load)} = \frac{PL}{4}$$

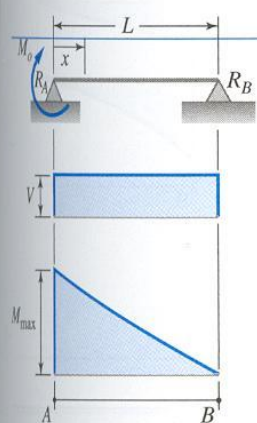
$$M_x \left(\text{when } x < \frac{L}{2}\right) = \frac{Px}{2}$$

$$y_{max} \text{ (at point of load)} = \frac{PL^3}{48EI}$$

$$y_x \left(\text{when } x < \frac{L}{2}\right) = \frac{Px}{48EI}(3L^2 - 4x^2)$$

$$\theta_A = -\theta_C = \frac{PL^2}{16EI}$$

Case 4. Simple Beam; End Couple M_0



$$R_A = V = -R_B = -\frac{M_0}{L}$$

$$V_x = R_A$$

$$M_{max} \text{ (at A)} = M_0$$

$$M_x = M_0 + R_A x$$

$$y_{max} \text{ (at } x = 0.422L) = 0.0642\frac{M_0 L^2}{EI}$$

$$y_x = \frac{M_0}{6EI}\left(\frac{x^3}{L} + 2Lx - 3x^2\right)$$

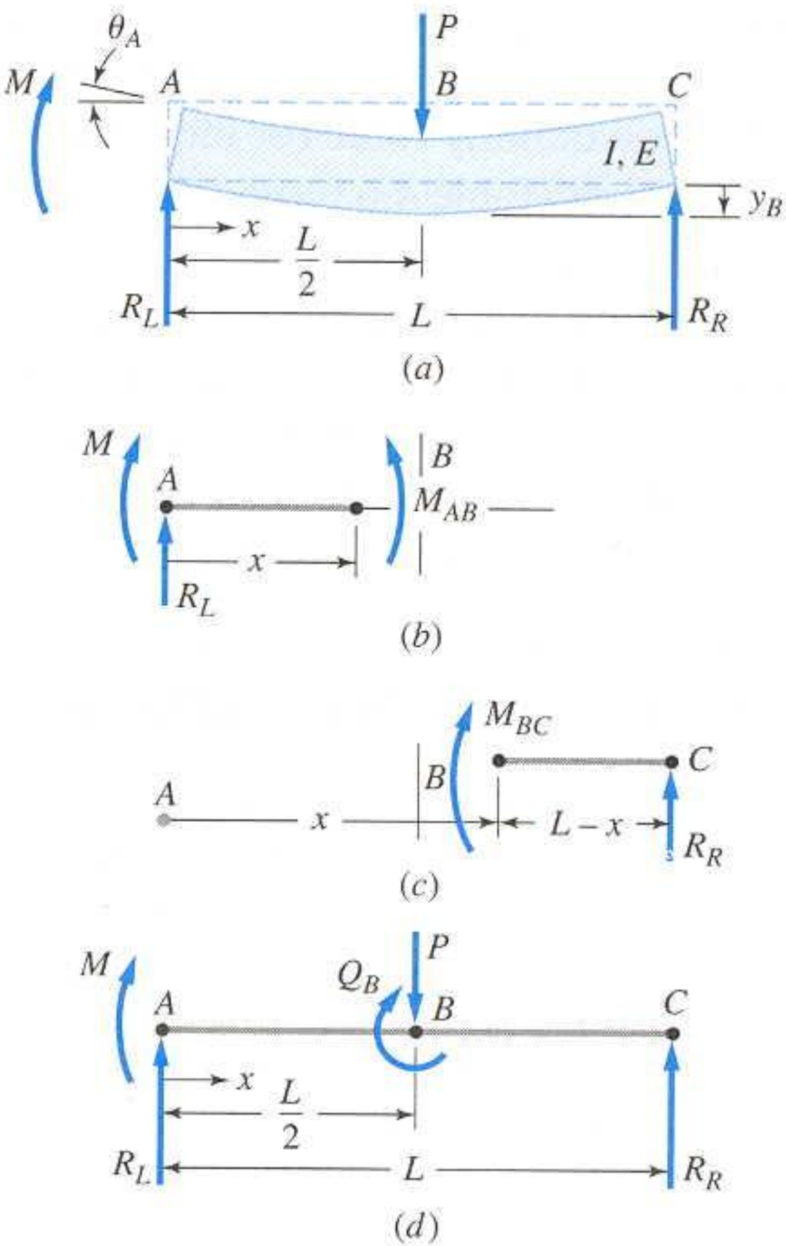
$$\theta_A = \frac{M_0 L}{3EI}$$

$$\theta_B = \frac{M_0 L}{6EI}$$

$$\theta_{midspan} \text{ (at } x = L/2) = -\frac{M_0 L}{24EI}$$

Figure E4.7

Simply supported beam with midspan load P and moment M at the left support.



Given: L_1=10 in, L_2=5 in, P=1000 lb, s=1.25 in, d=1.25 in

Find: Use Castigliano's theorem to find deflection y_0 under load "P"

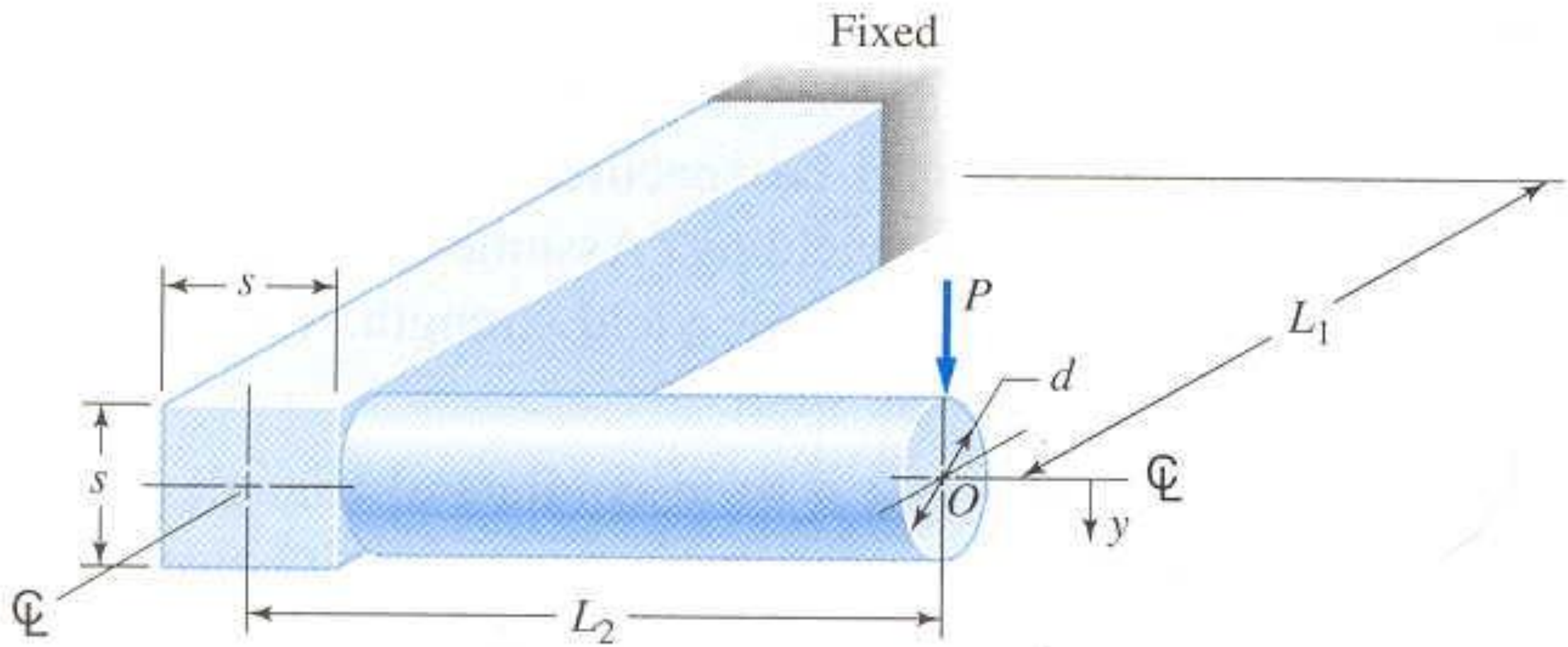


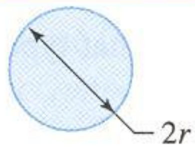
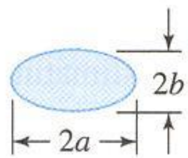
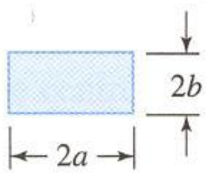
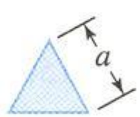
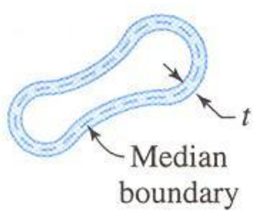
Figure P4.24

Right-angle support bracket with transverse end-load.

Summary of Textbook Problems – Problem 4.24, Castigliano's Theorem

- **Select coordinate system and identify the elements of the total strain energy**
  - Bending of square leg of support bracket
  - Torsion of square leg of support bracket
  - Bending of the round leg of support bracket
- **Differentiate the expression of "U" with respect to load "P", to find deflection y_0**
  - Stiffness properties of steel: $E=30x10^6$psi, $G=11.5x10^6$psi
  - Use Case 3 in Table 4.4 to find the geometric rigidity parameter, K, of the square leg, $K_1= 0.34$
  - Substitute all numerical values to obtain: y_0 = 0.13 inches

TABLE 4.4 Parameters for Determining Shearing Stress and Angular Deformation for Bars of Various Cross-Sectional Shape Subjected to Torsional Moments

Shape and Dimensions of Cross Section		Q, in^3	K, in^4	Location of Maximum Shearing Stress
1. Solid circular		$\dfrac{\pi r^3}{2}$	$\dfrac{\pi r^4}{2}$	On entire boundary
2. Solid elliptical		$\dfrac{\pi a b^2}{2}$	$\dfrac{\pi a^3 b^3}{a^2 + b^2}$	At ends of minor axis
3. Solid rectangular		$\dfrac{8a^2b^2}{3a + 1.8b}$ (for $a \geq b$)	$ab^3\left[\dfrac{16}{3} - 3.36\dfrac{b}{a}\left(1 - \dfrac{b^4}{12a^4}\right)\right]$ (for $a \geq b$)	At midpoint of each longer side
4. Solid equilateral triangle		$\dfrac{a^3}{20}$	$\dfrac{\sqrt{3}\,a^4}{80}$	At midpoint of each side
5. Any thin tube of uniform thickness t U = length of median boundary. A = area enclosed by median boundary.		$2At$	$\dfrac{4A^2t}{U}$	Nearly uniform throughout